Circular Strings and Multi-Strings in de Sitter and Anti de Sitter Spacetimes[1]

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Abstract

The exact general solution of circular strings in $2+1$ dimensional de Sitter spacetime is described completely in terms of elliptic functions. The novel feature here is that one single world-sheet generically describes *infinitely many* (different and independent) strings. This has no analogue in flat spacetime. The circular strings are either oscillating ("stable") or indefinitely expanding ("unstable"). We then compute the *exact* equation of state of circular strings in the $2+1$ dimensional de Sitter (dS) and anti de Sitter (AdS) spacetimes, and analyze its properties for the different (oscillating, contracting and expanding) strings. We finally perform a semi-classical quantization of the oscillating circular strings. The string mass is $m = \sqrt{C}/(\pi H\alpha')$, C being the Casimir operator, $C = -L_{\mu\nu}L^{\mu\nu}$, of the $O(3,1)$-dS [$O(2,2)$-AdS] group, and H is the Hubble constant. We find the mass formula $\alpha' m^2_{\text{dS}} \approx 4n - 5H^2\alpha' n^2$, $(n \in N_0)$, and a *finite* number of states $N_{\text{dS}} \approx 0.34/(H^2\alpha')$ in de Sitter spacetime; $m^2_{\text{AdS}} \approx H^2 n^2$ (large $n \in N$) and $N_{\text{AdS}} = \infty$ in anti de Sitter spacetime. The level spacing grows with n in AdS spacetime, while is approximately constant (although smaller than in Minkowski spacetime and slightly decreasing) in dS spacetime.

1 Introduction

The study of string dynamics in curved spacetimes reveals new insights and new physical phenomena with respect to string propagation in flat spacetime (and with respect to quantum fields in curved spacetimes) [1, 2, 3]. The results of this programme are relevant both for fundamental (quantum) strings and for cosmic strings, which behave essentially in a classical way.

Among the cosmological backgrounds of interest, de Sitter spacetime occupies a special place. It is on one hand relevant for inflation, and on the other hand, string propagation turns out to be specially interesting there. String-instability, in the sense that the string proper length grows indefinitely (proportional to the expansion factor of the universe) is particularly present in de Sitter spacetime [1-8]. Moreover, a novel feature for strings in curved spacetimes was first found in de Sitter spacetime: Exact *multi-string* solutions [6, 7]. That is, one single world-sheet generically describes two strings [7], several strings [6] or even *infinitely many* (different and independent) strings [8]

Circular strings are specially suited for detailed investigation. Since the string equations of motion become separable, one has to deal with non-linear ordinary differential equations instead of non-linear partial differential equations. In order to obtain generic non-circular string solutions the full power of the inverse scattering method is needed in de Sitter spacetime [6].

Here we will present some more recent results [8, 9] on exact circular string solutions in de Sitter spacetime and, for comparison, in anti de Sitter spacetime too. That is, we consider the circular string solutions of the equations of motion and constraints:

$$\ddot{x}^\mu - x''^\mu + \Gamma^\mu_{\rho\sigma}(\dot{x}^\rho \dot{x}^\sigma - x'^\rho x'^\sigma) = 0, \tag{1.1}$$

$$g_{\mu\nu}\dot{x}^\mu x'^\nu = g_{\mu\nu}(\dot{x}^\mu \dot{x}^\nu + x'^\mu x'^\nu) = 0, \tag{1.2}$$

where dot and prime stand for derivative with respect to the world-sheet coordinates τ and σ, respectively and $\Gamma^\mu_{\rho\sigma}$ are the Christoffel symbols with respect to the metric $g_{\mu\nu}$. The exact general solution of circular strings in $2+1$ dimensional de Sitter spacetime is described closely and completely in terms of elliptic functions (Section 2).

By computing the string energy and pressure, we obtain the corresponding equations of state, providing the physical interpretation of the solutions

in a cosmological context. We analyze the equations of state for the different (oscillating, contracting and expanding) strings. The string equation of state turns out to have the perfect fluid form $P = (\gamma - 1)E$, with the instantaneous coefficient γ depending on an elliptic modulus (Section 3).

Cosmological backgrounds like de Sitter and anti de Sitter spacetimes are not Ricci flat and hence they are not string vacua even at first order in α'. Strings are there non-critical and quantization will presumably lead to features like ghost states. No definite answer is available by now to such conformal anomaly effects.

It is important in this context to investigate the quantum aspects in the semi-classical regime, where anomaly effects are practically irrelevant. Semi-classical, in this context, means the regime in which $H^2\alpha' << 1$, where H is the Hubble constant. We semi-classically quantize the time-periodic string solutions in de Sitter and anti de Sitter spacetimes. Time-periodic string solutions here include all the circular string solutions in anti de Sitter spacetime, as well as the oscillating string solutions in de Sitter spacetime (Section 4).

2 Circular Strings in de Sitter Spacetime, Multi-Strings

Recently, several progresses in the understanding of string propagation in de Sitter spacetime have been obtained [6-11]. The classical string equations of motion (plus the string constraints) were shown to be integrable in D-dimensional de Sitter spacetime [10, 11]. They are equivalent to a non-linear sigma model on the Grassmannian $SO(D,1)/O(D)$ with periodic boundary conditions (for a closed string). In addition, the string constraints imply a zero world-sheet energy-momentum tensor, and these constraints are compatible with the integrability. Moreover, the exact string dynamics in de Sitter spacetime is equivalent to a generalized sh-Gordon model with a potential unbounded from below [11]. The sh-Gordon function $\alpha(\sigma,\tau)$ has here a clear physical meaning: $H^{-1}\exp[\alpha(\sigma,\tau)/2]$ determines the proper size of the string (H is the Hubble constant). In $2+1$ dimensions, the string dynamics is exactly described by the standard sh-Gordon equation.

More recently, a novel feature for strings in de Sitter spacetime was found:

Exact multi-string solutions [6, 7]. Exact circular string solutions were found describing two different strings [7]. One string is stable (the proper size is bounded), and the other one is unstable (the proper size blows up) for large de Sitter radius. Soliton methods, (the so-called "dressing method" in soliton theory) were implemented using the linear problem (Lax pair) of this system, in order to construct systematically exact string solutions [6]. The one-soliton string solution constructed in this way, generically describe five different and independent strings: one stable string and four unstable strings. These solutions (even the stable string) do not oscillate in time.

In this section, we go further in the investigation of exact string solutions in de Sitter spacetime. We find exact string solutions describing *infinitely many* different and independent strings. The novel feature here is that we have one single world-sheet but multiple (infinitely many) strings. The world-sheet time τ turns out to be an infinite-valued function of the target space time X^0 (which can be the hyperboloid time q^0, the comoving time T or the static coordinate time t). Each branch of τ as a function of q^0 corresponds to a different string. In flat spacetime, multiple string solutions are necessarily described my multiple world-sheets. Here, a single world-sheet describes infinitely many different and simultaneous strings as a consequence of the coupling to the spacetime geometry. These strings do not interact among themselves; all the interaction is with the curved spacetime.

We apply the circular string *Ansatz*:

$$t = t(\tau), \quad \phi = \sigma, \quad r = r(\tau), \tag{2.1}$$

in $2+1$ dimensional de Sitter spacetime, particularly convenient in terms of the static de Sitter coordinates (t, r, ϕ) (we also describe the solutions in the hyperboloid and comoving parametrizations). The string equations of motion and constraints, Eqs.(1.1)-(1.2), can be solved directly and completely in terms of elliptic functions. They reduce to two decoupled first order differential equations for the time component $t(\tau)$ and the string radius $r(\tau)$:

$$\dot{t} = \frac{\sqrt{b}\alpha'}{1 - H^2 r^2} \tag{2.2}$$

$$\dot{r}^2 + V(r^2) = b\alpha'^2; \quad V(r^2) = r^2(1 - H^2 r^2) \tag{2.3}$$

Notice that we are using the notation of Ref.[9], which is slightly different from the notation of Ref.[8]. The $\dot{r}$-equation is solved by:

$$H^2 r^2(\tau) = \wp(\tau - \tau_o) + 1/3, \tag{2.4}$$

where $\wp$ is the Weierstrass elliptic function [12] with discriminant $\Delta = 16b^2 H^4 \alpha'^4 (1 - 4bH^2\alpha'^2)$, and b and τ_o are integration constants (τ_o is generally complex and must be chosen such that $r(\tau)$ is real for real τ). The solutions depend on one constant parameter b (for fixed H, α') related to the string energy, see Section 3, and fall into three classes, depending on whether:

$$\begin{aligned} &bH^2\alpha'^2 < 1/4 \; (\Delta > 0), \\ &bH^2\alpha'^2 = 1/4 \; (\Delta = 0), \\ &bH^2\alpha'^2 > 1/4 \; (\Delta < 0). \end{aligned}$$

As can be seen in the diagram $(r^2, V(r^2))$, Fig.1., in which the full string dynamics takes place, these cases correspond to either oscillatory motion or infinite (unbounded) motion.

The proper string size S of the circular strings are given for all r by:

$$S(\tau) = r(\tau), \tag{2.5}$$

as can be seen from the induced line element on the string world-sheet:

$$ds^2 = r^2(\tau)(-d\tau^2 + d\sigma^2). \tag{2.6}$$

In the $bH^2\alpha'^2 = 1/4$ case, the Weierstrass elliptic function degenerates into a hyperbolic function:

$$H^2 r^2(\tau) = \frac{1}{2}[1 + \sinh^{-2}(\frac{\tau - \tau_o}{\sqrt{2}})]. \tag{2.7}$$

Two real independent solutions appear for the choices $\tau_o = i\pi/2$ and $\tau_o = 0$, respectively:

$$H^2 r_\pm^2(\tau) = \frac{1}{2}[\tanh(\frac{\tau}{\sqrt{2}})]^{\pm 2}. \tag{2.8}$$

(They were first found in Ref.[7]). We have also the solution $H^2 r_0^2 = 1/2$, corresponding to a stable string with constant proper size $S_0 = 1/(\sqrt{2}H)$

(i.e., sh-Gordon function $\alpha = 0$). This solution was found in Ref.[7] and we shall not discuss it here.

The solution r_- describes two different strings, I and II, as it can be seen from the hyperboloid time $q_-^0(\tau)$, Fig.2. Here τ is a two-valued function of q_-^0: String I corresponds to $-\infty < \tau < 0$ and string II to $0 < \tau < \infty$. The proper size S_- for both strings is given by Eq.(2.5), using Eq.(2.8). For $q_-^0 \to \infty$, string I is unstable, while string II is stable. The proper size $S_-(q_-^0 \to \infty)$ blows up for string I (for which $q_-^0 \to \infty$ corresponds to $\tau \to 0_-$), while tends to a constant value for string II (for which $q_-^0 \to \infty$ corresponds to $\tau \to \infty$). String I starts with minimal size $S_- = 1/(\sqrt{2}H)$ at $\tau = -\infty$ and blows up at $\tau = 0$. String II starts with infinite size at $\tau = 0$ but approaches $S_- = 1/(\sqrt{2}H)$ for $\tau \to \infty$.

The solution r_+ of this $bH^2\alpha'^2 = 1/4$ case describes only one stable string. The hyperboloid time $q_-^0(\tau)$, is a monotonically increasing function of τ and the corresponding string has bounded proper size S_+. The string starts with $S_+ = 1/(\sqrt{2}H)$, at $q_+^0 = -\infty$, it contracts until it collapses ($S_+ = 0$), then it expands until it reaches the original size for $q_+^0 = \infty$. For $bH^2\alpha'^2 = 1/4$ the evolution is always non-oscillatory. Even the stable string does not oscillate in time.

For $bH^2\alpha'^2 < 1/4$ there exist two real independent solutions for the choices $\tau_o = 0$ and $\tau_o = \omega'$, where ω' is the imaginary semi-period of the Weierstrass function:

$$H^2 r_-^2(\tau) = \wp(\tau) + 1/3, \tag{2.9}$$

$$H^2 r_+^2(\tau) = \wp(\tau + \omega') + 1/3. \tag{2.10}$$

r_- and r_+ are oscillating solutions as functions of τ. The solution r_- describes infinitely many strings; r_- has infinitely many branches $[0, 2\omega]$, $[2\omega, 4\omega]$, ..., each of which corresponds to a different string (ω is the real semi-period, of the Weierstrass function). This can be seen from the hyperboloid time $q_-^0(\tau)$, Fig.3: The world-sheet time τ is an infinite-valued function of q_-^0. The hyperboloid time q_-^0 blows up at the boundaries of the branches $\tau = \pm 2N\omega$ (N being an integer):

$$\mid q_-^0(\tau) \mid \sim \frac{1}{\mid 2N\omega - \tau \mid}. \tag{2.11}$$

Further insight is obtained by considering the comoving time T_- and the static coordinate time t_-. Closed expressions for them are given in Ref.[8],

in terms of Weierstrass ζ and σ-functions, and also rewritten in terms of elliptic theta-functions. The cosmic time T_- is singular at $\tau = 0$, $\tau = x/\mu$, $\tau = 2\omega$ and similarly in the other branches. (x, μ are two real constants. x is expressed as an incomplete elliptic integral of first kind while $\mu = \sqrt{(1+\sqrt{1-4bH^2\alpha'^2})/2}$). The static coordinate time t_-, on the other hand, is regular at the boundaries of the branches, but is singular at $\mu\tau = 2KN \pm x$:

$$t_-(\tau) \sim \frac{1}{2\pi} \log \mid \mu\tau - 2KN \mp x \mid, \tag{2.12}$$

where K is a complete elliptic integral of the first kind. It must be noticed that although r_- is periodic in τ, the comoving time is not, i.e. $T_-(\tau) \neq T_-(\tau + 2\omega)$. This implies that the infinitely many strings are different. The difference in proper size of the n'th and the $(n+1)$'th string for a given comoving time T_- is given by:

$$\Delta S_- = \frac{\pi}{H} \frac{\theta_1'}{\theta_1}(\frac{\pi x}{2K}) e^{HT_-} e^{-n\pi \frac{\theta_1'}{\theta_1}(\frac{\pi x}{2K})}. \tag{2.13}$$

However, *all* the strings are of the same type: *unstable*. For instance, in the branch $\tau \in [0, 2\omega]$, the string starts at $\tau = 0$ ($q_-^0 = -\infty$) with infinite size, then contracts to the minimal size $HS_- = \sqrt{(1+\sqrt{1-4bH^2\alpha'^2})/2}$ and eventually expands towards infinite size at $\tau = 2\omega$ ($q_-^0 = \infty$). These solutions never collapse to $r = 0$.

For the solution r_+ of the $bH^2\alpha'^2 < 1/4$ case, the string dynamics takes place inside the horizon. r_+, being a regularly oscillating function of τ, is then also a regularly oscillating function of the physical times q_+^0, T_+ and t_+. The static coordinate time t_+, from which one easily deduces q_+^0 and T_+, is expressed in terms of theta-functions [8]. The solution r_+ describes one stable string oscillating between its minimal size $S_+ = 0$ (collapse) and its maximal size $HS_+ = \sqrt{(1-\sqrt{1-4bH^2\alpha'^2})/2}$. It must be noticed that the string oscillations here do not follow a pure harmonic motion as in flat Minkowski spacetime, but they are precise superpositions of all frequencies $(2n-1)\Omega$, $(\Omega = \pi\mu/(2K),\ n = 1, 2, ..., \infty)$ [8]; here the non-linearity of the string equations of motion fixes the relation between the mode coefficients, and the basic frequency Ω depends on the string energy via b.

For $bH^2\alpha'^2 > 1/4$ two real independent solutions are obtained for $\tau_o = 0$ and $\tau_o = \omega_2'$, where ω_2' is the imaginary semi-period of the Weierstrass function:

$$H^2 r_-^2(\tau) = \wp(\tau) + 1/3, \tag{2.14}$$

$$H^2 r_+^2(\tau) = \wp(\tau + \omega_2') + 1/3. \tag{2.15}$$

In this case r_- again describes infinitely many strings, all of them are unstable. The difference with the $bH^2\alpha'^2 < 1/4$ case, is that here the strings have a collapse during their evolution. For instance, in the branch $\tau \in [0, 2\omega_2]$, where ω_2 is the real semi-period of the Weierstrass function, the string starts with infinite size at $\tau = 0$ ($q_-^0 = -\infty$), it then contracts until it collapses to a point and then it expands towards infinite size again at $\tau = 2\omega_2$ ($q_-^0 = \infty$). In contrast to the $bH^2\alpha'^2 < 1/4$ case, the solution r_+ is here just a time translated version of r_-:

$$r_+^2(\tau) = r_-^2(\tau + \omega_2), \tag{2.16}$$

and describes therefore essentially the same features as the solution r_-.

A summary of the main features and conclusions of the results of this section is presented in Table 1. Further details can be found in Ref.[8].

3 Physical Interpretation, Energy, Pressure

In this section we discuss the physical properties (energy, pressure) of a gas of circular strings in de Sitter and anti de Sitter spacetimes. In static coordinates, we thus consider the spacetimes:

$$ds^2 = -a(r)dt^2 + \frac{dr^2}{a(r)} + r^2 d\phi^2. \tag{3.1}$$

For simplicity we consider the string dynamics in a 2+1 dimensional spacetime. All our solutions can however be embedded in a higher dimensional spacetime, where they will describe plane circular strings. We are interested in the three cases:

$$\begin{aligned} a(r) &= 1 \quad \text{for Minkowski spacetime,} \\ a(r) &= 1 - H^2r^2 \quad \text{for de Sitter spacetime,} \\ a(r) &= 1 + H^2r^2 \quad \text{for anti de Sitter spacetime.} \end{aligned}$$

The equations describing the evolution of circular strings are:

$$\dot{t} = \frac{\sqrt{b}\alpha'}{a(r)} \tag{3.2}$$

$$\dot{r}^2 + V(r) = b\alpha'^2; \quad V(r) = r^2 a(r) \tag{3.3}$$

and generalize Eqs.(2.2)-(2.3).

Properties like energy and pressure of the strings are more conveniently discussed in comoving (cosmological) coordinates:

$$ds^2 = -(dT)^2 + a^2(T)\, \frac{dR^2 + R^2\, d\phi^2}{(1 + \frac{k}{4}R^2)^2}, \tag{3.4}$$

including as special cases Minkowski, de Sitter and anti de Sitter spacetimes:

$$\begin{aligned} a(T) &= 1, \;\; k=0 \quad \text{for Minkowski spacetime,} \\ a(T) &= e^{HT}, \;\; k=0 \quad \text{for de Sitter spacetime,} \\ a(T) &= \cos HT, \;\; k=-H^2 \quad \text{for anti de Sitter spacetime.} \end{aligned}$$

The spacetime energy-momentum tensor is given in $2+1$ dimensions by:

$$\sqrt{-g}\, T^{\mu\nu} = \frac{1}{2\pi\alpha'} \int d\tau d\sigma\, (\dot{X}^\mu \dot{X}^\nu - X'^\mu X'^\nu)\, \delta^{(3)}(X - X(\tau,\sigma)). \tag{3.5}$$

After integration over a spatial volume that completely encloses the string [13], the energy-momentum tensor for a circular string takes the form of a fluid:

$$T^\mu_\nu = \text{diag.}(-E,\; P,\; P), \tag{3.6}$$

where, in the comoving coordinates (3.4):

$$E(X) = \frac{1}{\alpha'}\, \dot{T}, \tag{3.7}$$

$$P(X) = \frac{1}{2\alpha'}\, \frac{a^2(T)}{(1 + \frac{k}{4}R^2)^2}\, \frac{\dot{R}^2 - R^2}{\dot{T}}, \tag{3.8}$$

represent the string energy and pressure, respectively.

We have already described all the circular string solutions in de Sitter spacetime, Section 2. The circular string solutions in anti de Sitter spacetime were obtained in Ref.[14]. The general solution is:

$$Hr(\tau) = \frac{\bar{k}}{\sqrt{1-2\bar{k}^2}}\mathrm{cn}[\frac{\tau}{\sqrt{1-2\bar{k}^2}},\ \bar{k}], \tag{3.9}$$

where:

$$\bar{k}^2 = \frac{-1+\sqrt{1+4bH^2\alpha'^2}}{\sqrt{1+4bH^2\alpha'^2}}. \tag{3.10}$$

For arbitrary values of the elliptic modulus $\bar{k} \in [0,\ \sqrt{1/2}\ [$, the solution describes one oscillating string.

Let us now return to the energy-momentum tensor for circular strings. The circular string solutions in de Sitter and anti de Sitter spacetimes depend on the elliptic modulus $k \equiv \sqrt{1-\mu^2}\ /\mu$ and $\bar{k}$, respectively, as we have seen. From these exact solutions we find the corresponding energy-momentum tensors. They turn out to have the perfect fluid form with an equation of state:

$$P = (\gamma - 1)E, \tag{3.11}$$

where γ in general is time-dependent and depends on the elliptic modulus (k or $\bar{k}$) as well. We have analyzed the equation of state, Eq.(3.11), for all circular string solutions in de Sitter and anti de Sitter spacetimes, Ref.[9].

In de Sitter spacetime, for strings expanding from zero radius towards infinity, the equation of state changes continuosly from the ultra-relativistic matter-type $P = +E/2$, (in $2+1$ dimensions) when $r \approx 0$, to the unstable string-type [4, 5], $P = -E/2$, when $r \to \infty$, see Fig.4.

On the other hand, for an oscillating stable string in de Sitter spacetime, γ oscillates between $\gamma(r=0) = 3/2$ and $\gamma(r = r_{\mathrm{max}}) = 1/2 + k^2/(1+k^2)$, where $k \in [0,\ 1]$. Averaging over one oscillation period, the pressure *vanishes* [9]. That is, these stable string solutions actually describe (in average) *cold matter*. See also Fig.5.

In anti de Sitter spacetime, only the oscillating (stable) circular string solutions, Eq.(3.9), exist. We find that γ oscillates between $\gamma(r=0) = 3/2$ and $\gamma(r = r_{\mathrm{max}}) = 1/2$, i.e. the equation of state "oscillates" between $P = +E/2$ and $P = -E/2$. This is similar to the situation in flat Minkowski spacetime. When averaging over an oscillation period in anti de Sitter spacetime, we find that γ takes values from 1 to $1 + 1/\pi^2$ for the allowed range

of the elliptic modulus. That is, the average pressure over one oscillation period is always *positive* in anti de Sitter spacetime.

In conclusion, positive pressure characterizes the regime in which the string radius is small relative to the string maximal size, while negative pressure is characteristic for the regime in which the string radius is large. In Minkowski spacetime, the two regimes are of equal "size", in the sense that the average pressure is identically zero. The influence of the spacetime curvature is among other effects, to modify the relative "size" of the two regimes.

A summary of these features are presented in Table 2. More details are given in Ref.[9].

4 Semi-Classical Quantization

In this section we perform a semi-classical quantization of the circular string configurations discussed in the previous sections. We use an approach developed in field theory by Dashen et. al. [15, 16], based on the stationary phase approximation of the partition function. The method can be only used for time-periodic solutions of the classical equations of motion. In our string problem, these solutions however, include all the circular string solutions in Minkowski and in anti de Sitter spacetimes, as well as the oscillating circular strings ($bH^2\alpha'^2 \leq 1/4$, c.f. Section 2) in de Sitter spacetime.

The result of the stationary phase integration is expressed in terms of the function:

$$W(m) \equiv S_{\text{cl}}(T(m)) + m\, T(m), \tag{4.1}$$

where S_{cl} is the action of the classical solution, m is the mass and the period $T(m)$ is implicitly given by:

$$\frac{dS_{\text{cl}}}{dT} = -m. \tag{4.2}$$

In string theory we must choose T to be the period in a physical time variable. For example, when a light cone gauge exists, T is the period in $X^0 = \alpha' p\tau$. The bound state quantization condition then becomes [15, 16]:

$$W(m) = 2\pi n, \quad n \in N_0 \tag{4.3}$$

for n 'large'. The method has been successfully used in many cases from quantum mechanics to quantum field theory. For integrable field theories the

semi-classical quantization happens in fact, to be exact. It must be noticed that string theory in de Sitter spacetime is exactly integrable [10, 11].

The actual computation of $W(m)$ is straightforward but tedious; the details are given in Ref.[9].

For completeness we first considered flat Minkowski spacetime. In that case, Eq.(4.3) reproduces the exact mass spectrum, except for the intercept. That is, the mass spectrum becomes $\alpha' m^2 = 4\, n, \quad n \in N_0$.

We find for de Sitter (anti de Sitter) spacetime that the mass is exactly proportional to the square-root of the Casimir operator $C = -L_{\mu\nu}L^{\mu\nu}$ of the $O(3,1)$-de Sitter [$O(2,2)$-anti de Sitter] group:

$$m = \frac{\sqrt{C}}{\pi H \alpha'}. \tag{4.4}$$

In Fig.6 we give parametric plots of $H^2\alpha' W$ as a function of $H^2 m^2 \alpha'^2$ for $k \in [0,\ 1]$ for de Sitter spacetime and for $\bar{k} \in [0,\ 1/\sqrt{2}\ [$ for anti de Sitter spacetime, respectively. We find for de Sitter spacetime:

$$\alpha' m^2_{\mathrm{dS}} \approx 4\, n - 5H^2\alpha'\, n^2, \quad n \in N_0 \tag{4.5}$$

This is different from the mass spectrum in Minkowski spacetime. The level spacing is however still approximately constant, but the levels are less separated than in Minkowski spacetime. Notice in particular that the level spacing slightly decreases for larger and larger n. In de Sitter spacetime there is only a *finite* number of levels as can be seen from Fig.1. The number of quantized circular string states can be estimated to be:

$$N_q \approx \frac{0.34}{H^2\alpha'}. \tag{4.6}$$

It is interesting to compare this result with the number of particle states obtained using canonical quantization [1]. One finds in this way a maximum number of states:

$$N_{\mathrm{max}} \approx \frac{0.15}{H^2\alpha'}, \tag{4.7}$$

which is of the same order as the semi-classical value Eq.(4.6). It must be noticed that in de Sitter spacetime, these states can *decay* quantum mechanically due to the possibility of quantum mechanical tunneling through the potential barrier, see Fig.1. Semi-classically, the decay probability is

however highly suppressed for $H^2\alpha' << 1$ and for any value of the elliptic modulus k, except near $k = 1$ where the barrier disappears, and for which the tunneling probability is close to unity, see Ref.[9].

In anti de Sitter spacetime arbitrary high mass states exist. The quantization of the high mass states yields (see Ref.[9]):

$$\alpha' m^2_{\text{AdS}} \approx H^2\alpha'\, n^2. \tag{4.8}$$

Thus the (mass)^2 grows like n^2 and the level spacing grows proportionally to n. This is a completely different behaviour as compared to Minkowski spacetime where the level spacing is constant. A similar result was recently found, using canonical quantization of generic strings in anti de Sitter spacetime [14, 17]. The physical consequences, especially the non-existence of a critical string temperature (Hagedorn temperature), of this kind of behaviour is discussed in detail in Ref.[17].

For both de Sitter and anti de Sitter spacetimes we find thus a strong qualitative agreement between the results obtained using canonical quantization, based on generic string solutions (string perturbation approach), and the results obtained using the semi-classical approach, based on oscillating circular string configurations.

A summary of these features are presented in Table 3. More details are given in Ref.[9].

5 Conclusion

We have found the exact general evolution of circular strings in $2+1$ dimensional de Sitter spacetime. We have expressed it closely and completely in terms of elliptic functions, see Ref.[8] for the details. The solution generically describes infinitely many (different and independent) strings, and depends on one constant parameter b (for fixed H, α') related to the string energy. We have computed *exactly* the equation of state of the circular string solutions in de Sitter and anti de Sitter spacetimes. The string equation of state has the perfect fluid form $P = (\gamma - 1)E$, with P and E expressed closely and completely in terms of elliptic functions (see Ref.[9] for the details) and the instantaneous parameter γ depending on an elliptic modulus. We have quantized the time-periodic (oscillating) string solutions within the semi-classical (stationary phase approximation) approach and found the mass formulas in

de Sitter and anti de Sitter spacetimes. The semi-classical quantization of the *exact* circular string solutions and the canonical quantization of generic strings (string perturbation series approach), provide the same qualitative results.

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Figure Captions

Figure 1. The potential $V(r^2) = r^2(1 - H^2r^2)$, defined in Eq.(2.3). For $bH^2\alpha'^2 < 1/4$, it acts effectively as a barrier. The horizon corresponds to $H^2r^2 = 1$.

Figure 2. The hyperboloid time q_-^0, in the $bH^2\alpha'^2 = 1/4$ case, as a function of τ. Notice that τ is a two-valued function of q_-^0.

Figure 3. (a) The hyperboloid time q_-^0 as a function of τ in the elliptic case $bH^2\alpha'^2 < 1/4$. Each of the infinitely many branches corresponds to one string. (b) The comoving time T_- as a function of τ in the elliptic case $bH^2\alpha'^2 < 1/4$. Notice that T_- is not periodic.

Figure 4. The energy and pressure for a string expanding from $r = 0$ towards infinity (unstable string) in de Sitter spacetime. The curves are drawn for the case $bH^2\alpha'^2 = 0.3$.

Figure 5. The energy and pressure for an oscillating (stable) string in de Sitter spacetime. The curves describe one period of oscillation. The curves are drawn for the case $bH^2\alpha'^2 = 0.15$.

Figure 6. (a) Parametric plot of $H^2\alpha'W$ as a function of $H^2m^2\alpha'^2$ for $k \in [0,\ 1]$ in de Sitter spacetime. There is only a finite number of states. (b) Parametric plot of $H^2\alpha'W$ as a function of $H^2m^2\alpha'^2$ in anti de Sitter spacetime. There are infinitely many states.

Table Captions

Table 1. Circular string evolution in de Sitter spacetime. For each value of $bH^2\alpha'^2$, there exists two independent solutions r_- and r_+:

Table 2. Circular string energy and pressure in Minkowski, de Sitter and anti de Sitter spacetimes.

Table 3. Semi-classical quantization of oscillating circular strings in Minkowski, de Sitter and anti de Sitter spacetimes.